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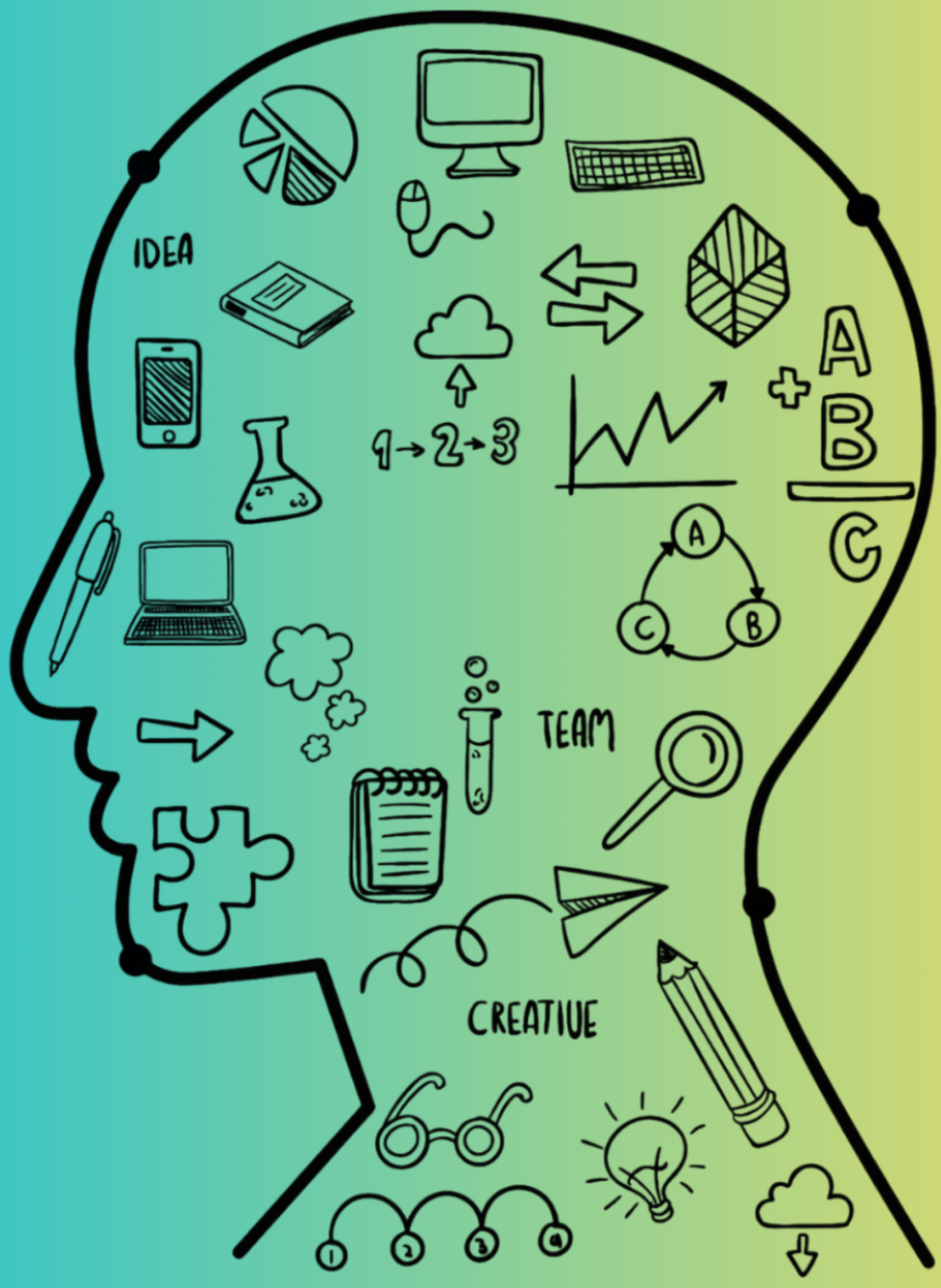
CALCULUS AND ITS APPLICATIONS

Mathematical Techniques



Written by:

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First Edition

Calculus and Its Applications

Mathematical Techniques

Bakti Siregar, M.Sc., CDS.

Table of contents

In the evolving landscape of science, engineering, and technology, calculus remains a fundamental tool for understanding change, modeling complex systems, and solving real-world problems. From the classical challenges of motion and geometry to modern applications in data science, optimization, and engineering, calculus provides a unifying language that bridges theory and practice. By mastering its core concepts and techniques, students and practitioners can analyze dynamic processes, make informed decisions, and develop models that foster innovation and excellence across disciplines.

This book, Calculus and Its Applications: Mathematical Techniques, offers a structured and comprehensive introduction to calculus. Beginning with the foundations of real numbers and the essentials of functions, readers are gradually guided through special functions, limits, and the core principles of derivatives. Building on these fundamentals, the text explores both the applications of derivatives in optimization and modeling, as well as the theory and practice of indefinite integrals and their wide-ranging applications. The journey concludes with a discussion of transcendental functions, connecting classical concepts to advanced and contemporary challenges.

Beyond theory, the book emphasizes practical applications—showing how calculus underpins decision-making, system optimization, and problem-solving in diverse fields. Each chapter integrates concepts with examples that reflect both traditional mathematical problems and modern technological contexts.

Through this approach, readers will not only develop a strong understanding of the mathematical principles of calculus but also gain the skills to apply them effectively to real-world challenges—fulfilling the book's vision of connecting classical problems with modern challenges.

Preface

About the Writer

Bakti Siregar, M.Sc., CDS works as a Lecturer at the ITSB Data Science Program. He earned his Master's degree from the Department of Applied Mathematics at National Sun Yat Sen University, Taiwan. In addition to teaching, Bakti also works as a Freelance Data Scientist for leading companies such as JNE, Samora Group, Pertamina, and PT. Green City Traffic.

He has a strong enthusiasm for projects (and teaching) in the fields of Big Data Analytics, Machine Learning, Optimization, and Time Series Analysis, particularly in finance and investment. His core expertise lies in statistical programming languages such as R Studio and Python. He is also experienced in implementing database systems like MySQL/NoSQL for data management and is proficient in using Big Data tools such as Spark and Hadoop.

Some of his projects can be viewed here: Rpubs, Github, Website, and Kaggle

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Acknowledgments

Calculus plays a vital role in modeling, analyzing, and optimizing processes across science, engineering, and technology. This book introduces fundamental concepts and techniques in calculus, including:

- A solid foundation in real numbers, functions, and limits
- The ability to analyze and interpret data across engineering and scientific contexts
- A clear understanding of the role of derivatives and integrals in modeling and problem-solving
- Practical skills in applying numerical methods and calculus techniques to real-world challenges

This book is designed for beginners seeking to build a strong foundation in calculus while appreciating its concepts and diverse applications—from classical mathematical problems to modern scientific and engineering challenges. We value the active participation of readers, whose insights and questions enrich the learning journey. It is our hope that this material serves not only as an introduction to calculus but also as a practical guide for applying mathematical reasoning to contemporary problems.

Feedback & Suggestions

Your feedback is invaluable in enhancing the quality of this book. We warmly invite readers to share their thoughts on the content, organization, and clarity of the material. Suggestions for additional topics, extended explanations, or further real-world applications are highly encouraged.

With your support and contributions, our goal is to make this book a comprehensive and accessible resource on calculus and its applications—from classical problems to modern challenges. Thank you for your engagement and feedback.

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Chapter 1

Introduction to Calculus

Calculus (Mathematical Techniques) is the mathematics of change and motion, offering powerful tools to model dynamic systems, solve complex problems, and make predictions in science, engineering, and technology. Mastery of calculus enables us to analyze diverse real-world phenomena, from the path of a moving object to the efficient allocation of resources.

The Figure 1.1 presents a visual overview of the chapter, highlighting the structure of key topics and their interconnections. It provides readers with a clear guide to navigate the material and understand how concepts link to applications.

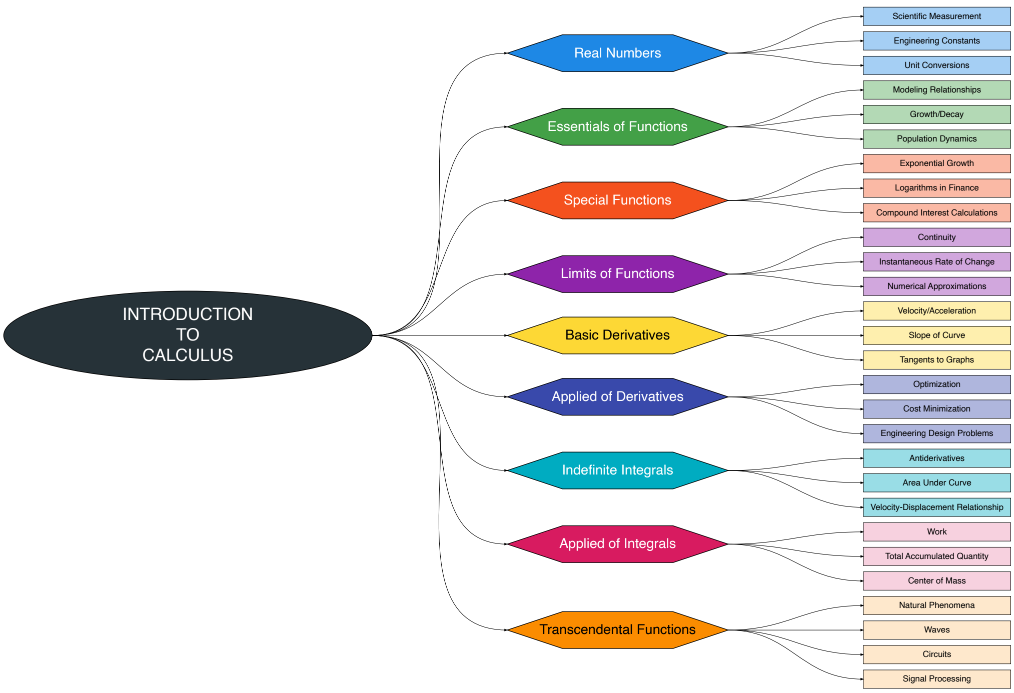

Figure 1.1: Mind Map of Introduction to Calculus

Table 1.1: Matrix Transformation

Property	Description	Example
Closure	Operations on real numbers yield real numbers	$a+b, a-b, a \cdot b, a/b$ (if $b \neq 0$)
Order	Real numbers can be compared and ordered	$a < b \rightarrow$ comparing magnitudes
Density	There is always another real number \ between two numbers	$\exists c : a < c < b \setminus \rightarrow$ interpolation
Absolute Value	Measures distance from zero	$\|a\| \rightarrow$ distance from zero
Scientific Measurement	Represent measurable physical quantities	$g = 9.8\,\mathrm{m/s}^2$
Engineering Constants	Constants used in formulas \ and modeling	E, $c = 3 \times 10^8\,\mathrm{m/s}$

This chapter introduces the fundamental building blocks of calculus, including real numbers, functions, limits, derivatives, integrals, and transcendental functions. Each concept is connected to practical applications to illustrate how calculus underpins real-world problem-solving.

1.1 Real Numbers

Real numbers are the **foundation of all calculus concepts**, forming the set of numbers that includes integers, fractions, and decimals. They are essential for performing calculations, defining functions, and understanding limits, derivatives, and integrals. A solid understanding of real numbers allows us to work with continuous quantities, measure physical phenomena, and apply mathematical reasoning in real-world contexts. The key properties, descriptions, and typical applications of real numbers are summarized in Table 1.1.

1.2 Essentials of Functions

Functions describe relationships between variables, showing how one quantity changes with another. They are fundamental in calculus as they provide **mathematical models** for dynamic systems, patterns, and processes across science, engineering, and economics. A proper understanding of functions requires knowledge of their **domain, range, types, and behavior**. An overview of the key concepts, descriptions, and applications is given in Table 1.2.

1.3 Special Functions

Special functions are widely used in science, engineering, and applied mathematics because they **model specific natural or engineered phenomena**. Extending beyond simple polynomials, they play a central role in solving real-world problems across

Table 1.2: Key Concepts of Functions

Key Concept	Description	Example / Application
Definition	Maps each x in domain to a unique y in range	$f : x \mapsto y$
Domain and Range	Domain = all possible inputs, Range = all outputs	$x \in [0, 10], f(x) \in [0, 100]$
Linear Function	Straight-line relationship	$f(x) = mx + b$
Quadratic Function	Parabolic relationship	$f(x) = ax^2 + bx + c$
Polynomial Function	Sum of powers of x	$f(x) = a_n x^n + \cdots + a_0$
Exponential Function	Rapid growth or decay	$P(t) = P_0 e^{rt}$ \ (population growth)
Trigonometric Function	Models periodic behavior	$T(t) = T_{avg} + A \sin\left(\frac{2\pi}{24} t\right)$ \ (temperature changes)

Table 1.3: Exponential, Logarithmic, and Trigonometric Functions

KeyConcept	Description	ExampleApplication
Exponential Functions	$f(x) = ae^{bx}$; models growth and decay	Compound interest: $A = Pe^{rt}$
Logarithmic Functions	$f(x) = \log_b(x)$; inverse of exponential	pH scale, sound intensity measurements
Trigonometric Functions	$f(x) = \sin x, \cos x, \tan x$; models periodic behavior	Pendulum motion: $\theta(t) = \theta_0 \cos(\omega t + \phi)$

physics, chemistry, biology, and finance. A structured overview of their main types, descriptions, and applications is provided in Table 1.3.

1.4 Limits of Functions

Limits reveal how functions **behave as inputs get closer to a given value**. They are essential in calculus, laying the groundwork for derivatives and continuity. With limits, we can explore instantaneous change, refine approximations, and resolve problems where direct evaluation fails. An overview of the main ideas and applications appears in Table 1.4.

Table 1.4: Key Concepts in Limits and Continuity

KeyConcept	Description	ExampleApplication
Definition	$\lim_{x\to a} f(x) = L$; the value $f(x)$ approaches as $x \to a$	Approximating function values: $\lim_{x\to 0} \frac{\sin x}{x} = 1$
One-sided Limits	Limits approaching from left $(x \to a^-)$ or right $(x \to a^+)$	Instantaneous velocity from left/right time intervals
Continuity	f is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$	Ensuring smooth motion or consistent output in physical systems

Table 1.5: Key Concepts in Derivatives

KeyConcept	Description	ExampleApplication
Definition	$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$; rate of change at x	Instantaneous velocity: $v(t) = s'(t)$
Interpretation	Slope of tangent line; instantaneous rate of change	Slope of a hill: $m = h'(x)$
Basic Rules	Power rule, sum rule, constant multiple rule	$\frac{d}{dx}x^n = nx^{n-1}$, $\frac{d}{dx}[f+g] = f' + g'$

Table 1.6: Applications of Derivatives

KeyConcept	Description	ExampleApplication
Optimization	Find maxima or minima by solving $f'(x) = 0$ \ and checking $f''(x)$	Maximize profit: $P'(x) = 0$
Rate of Change	Quantifies how one variable changes \ with respect to another	Velocity: $v(t) = s'(t)$
Critical Points	Points where $f'(x) = 0$ or undefined; \ used to find maxima, minima, or inflection points	Minimizing cost: $C'(x) = 0$; \ analyzing structure stress
Motion Analysis	Derivatives of position give \ velocity and acceleration	$v(t) = s'(t)$, $a(t) = s''(t)$

1.5 Basic Derivatives

Derivatives quantify how a function changes with respect to its input, **capturing slopes, rates of change, and tangent behavior**. They play a central role in calculus by enabling the analysis of motion, growth, optimization, and a wide range of dynamic processes. Key concepts, descriptions, and applications are summarized in Table 1.5.

1.6 Applied Derivatives

Applied derivatives illustrate how differentiation is used to **solve real-world problems** across engineering, economics, physics, and other fields. By examining rates of change, extrema, and concavity, derivatives provide tools for optimizing processes, predicting behavior, and supporting decision-making. Key concepts, descriptions, and applications are summarized in Table 1.6.

1.7 Indefinite Integrals

Indefinite integrals, or antiderivatives, **undo the process of differentiation**, enabling us to recover the original function from its derivative. They represent accumulated quantities such as displacement, total growth, or total charge. The Table 1.7 below summarizes key concepts, descriptions, and example applications:

Table 1.7: Indefinite Integrals

KeyConcept	Description	ExampleApplication
Definition	$F'(x) = f(x)$; $\int f(x)dx = F(x) + C$	Displacement: $s(t) = \int v(t)dt$; e.g., $v(t) = 3t^2 \implies s(t) = t^3 + C$
Power Rule	$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$	Integration of x^2 gives $\frac{x^3}{3} + C$
Constant Multiple	$\int cf(x)dx = c\int f(x)dx$	Multiply constant with integral
Sum Rule	$\int[f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$	$\int(x^2 + 2x)dx = \frac{x^3}{3} + x^2 + C$
Total Accumulation	Accumulation of quantity over time	Revenue: $\int R(t)dt$

Table 1.8: Definite Integrals

KeyConcept	Description	ExampleApplication
Definite Integral	Total accumulation of a quantity over interval $[a, b]$: $\int_a^b f(x)dx$	Area under curve: $\int_0^2 x^2 dx = \frac{8}{3}$
Area Under a Curve	Calculates area between function and x-axis	Same as above
Physical Applications	Integrals for work, mass, charge, revenue	Mass: $M = \int_a^b \rho(x)dx$, Work: $W = \int_a^b F(x)dx$

1.8 Applied Integrals

Transcendental functions are those that **cannot be represented as finite polynomials**, including **exponential, logarithmic, trigonometric, and inverse trigonometric functions**. They play a central role in mathematics, physics, engineering, and the applied sciences by modeling complex natural and engineered phenomena. Key types, descriptions, and example applications are summarized in Table 1.8.

1.9 Transcendental Functions

Transcendental functions are functions that **cannot be expressed as finite polynomials**. They include **exponential, logarithmic, trigonometric, and inverse trigonometric functions**, and are essential in advanced mathematics, physics, engineering, and applied sciences for modeling complex phenomena. Transcendental functions are used to model complex phenomena in science and engineering in the table Table 1.9.

Table 1.9: Special Functions

KeyConcept	Description	ExampleApplication
Exponential Functions	$f(x) = e^x$ or a^x, model growth and decay	RC circuit voltage: $V(t) = V_0(1 - e^{-t/RC})$
Logarithmic Functions	$f(x) = \ln x$ or $\log_a x$, used in scaling	Measuring pH, sound intensity
Trigonometric Functions	$f(x) = \sin x, \cos x, \tan x$, model periodic behavior	Wave motion: $y(x, t) = A\sin(kx - \omega t)$
Inverse Trigonometric Functions	$f(x) = \arcsin x, \arccos x, \arctan x$, solving angles	Population oscillations: $P(t) = P_{\text{avg}} + A\cos(\omega t + \phi)$

References

Chapter 2

Real Numbers

Understanding **Real Numbers** ($\mathbb{R}$) is the first step in exploring the world of **real analysis**. These numbers serve as the essential building blocks for calculus, algebra, numerical modeling, and various applied sciences. They provide a framework for representing quantities, measuring change, and describing continuous processes in both mathematics and real-world applications [1]–[3].

To help navigate the key aspects of real numbers, the Figure 2.1 offers a **5W+1H mind map**. This visualization guides learners through the **What**—definitions and subsets; the **Why**—their importance and significance; the **When**—historical discoveries and formalization; the **Where**—applications in science, engineering, economics, and daily life; the **Who**—mathematicians and everyday users; and the **How**—representation on the number line, decimal forms, and intervals. By following this map, one can see not just the numbers themselves, but their role and relevance across disciplines.

The following Table 2.1 presents a structured summary of the **5W+1H questions** related to Real Numbers, based on the Figure 2.1 mind map. It organizes the material into categories—**What, Why, When, Where, Who, How**—to guide learners in understanding the definitions, subsets, properties, number line representation, and applications of real numbers in science, engineering, economics, and daily life.

2.1 Definition

The real numbers ($\mathbb{R}$) are the set of numbers that include both rational numbers (fractions of integers) and irrational numbers (numbers that cannot be expressed as fractions). They can be represented on the number line, which extends infinitely in both positive and negative directions [1], [2].

Formally:

$$\mathbb{R} = \{x \mid x \text{ corresponds to a point on the number line}\}.$$

Table 2.1: 5W+1H Questions for Real Numbers

Question	Notes
WHAT?	
What are real numbers ($\mathbb{R}$)?	Definition of real numbers
What are the subsets of real numbers?	Natural, Whole, Integers, Rational, Irrational
What is the difference between rational and irrational numbers?	Rational can be expressed as p/q, irrational cannot
What are the key properties of real numbers?	Closure, Commutative, Associative, Distributive, Identity, Inverse
WHY?	
Why are real numbers important in mathematics?	Used in mathematics, science, engineering, daily life
Why do we need to know properties like closure and distributive law?	Ensures operations are valid within real numbers
Why are irrational numbers significant in science and engineering?	Irrational numbers appear in physics, geometry, engineering constants
WHEN?	
When were irrational numbers discovered?	Ancient Greek mathematicians discovered $\sqrt{2}$
When did mathematicians formalize the set of real numbers?	Formalized in 19th century by Dedekind and Cantor
When do we use real numbers in real-life applications?	Used in measurement, finance, statistics, and computations
WHERE?	
Where are real numbers applied in science and engineering?	Physics formulas, engineering measurements, chemistry calculations
Where can real numbers be observed in daily life?	Counting, money, time, distances
Where in mathematics do we need to distinguish rational and irrational numbers?	Algebra, number theory, functions
WHO?	
Who were the key mathematicians in the development of real numbers?	Euclid, Cantor, Dedekind
Who uses real numbers most frequently in practical applications?	Scientists, engineers, mathematicians
Who introduced the concept of number line and interval representation?	Mathematicians formalized number line representation
HOW?	
How are real numbers represented on the number line?	Graphically as points on the number line
How do we write rational numbers in decimal form?	Terminating, repeating, or non-repeating decimals
How do we express intervals using	Using [a,b] or (a,b) interval

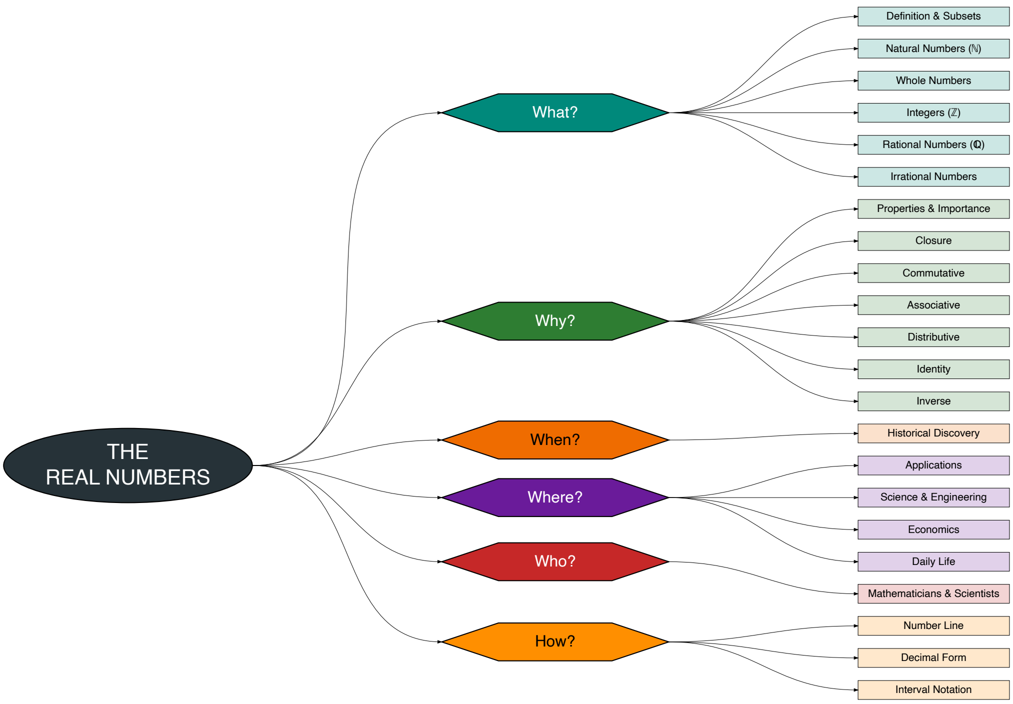

Figure 2.1: Real Numbers with 5W+1H Notes

2.2 Subsets of Real Numbers

Real numbers ($\mathbb{R}$) consist of several subsets, each with distinct properties and applications. Understanding these subsets is fundamental in mathematics, physics, and engineering.

2.2.1 Natural Numbers ($\mathbb{N}$)

Natural numbers are the set of positive counting numbers used for enumerating objects. Formally, the set of natural numbers is written as

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}.$$

Natural numbers have several important properties. They are always positive, and they are closed under addition and multiplication. However, they are not closed under subtraction or division; for example, $2 - 3 \notin \mathbb{N}$. Some examples of natural numbers include $1, 2, 3, 10, 100$, and so on. These numbers are widely used in everyday life and in mathematics for counting discrete objects, numbering sequences, and performing basic arithmetic operations.

2.2.2 Whole Numbers

Whole numbers extend natural numbers by including zero. Formally, the set of whole numbers is written as

$$\text{Whole Numbers} = \{0, 1, 2, 3, \dots\}.$$

Whole numbers have several important properties. They are non-negative and are closed under addition and multiplication. However, they are not closed under subtraction; for example, $0 - 1 \notin$ Whole Numbers. Some examples of whole numbers include $0, 1, 2, 50, 1000$, and so on. Whole numbers are widely used in numbering positions, indexing in programming, and counting objects when zero is included.

2.2.3 Integers ($\mathbb{Z}$)

Integers include all whole numbers and their negative counterparts. Formally, the set of integers is written as

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

Integers have several important properties. They are closed under addition, subtraction, and multiplication, but they are not closed under division; for example, $1/2 \notin \mathbb{Z}$. Some examples of integers include $-10, -1, 0, 3, 15$, and so on. Integers are widely used for representing gains and losses, elevations, temperatures, and positions relative to a reference point.

2.2.4 Rational Numbers ($\mathbb{Q}$)

Rational numbers are numbers that can be expressed as a fraction $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$. Formally, the set of rational numbers is written as

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

Rational numbers have several important properties. They can be positive, negative, or zero, and they are closed under addition, subtraction, multiplication, and division (except division by zero). They can also be represented as terminating or repeating decimals. Some examples of rational numbers include $\frac{1}{2}, -\frac{7}{3}, 0.75$, and $0.333\dots$. Rational numbers are widely used in fractions for measurements, probabilities, ratios, and proportional relationships.

2.2.5 Irrational Numbers

Irrational numbers cannot be expressed as a fraction $\frac{p}{q}$ with integers p and q, and their decimal expansions are non-terminating and non-repeating. They can be positive or negative and are generally closed under addition, subtraction, multiplication, and sometimes division, but they cannot be represented exactly as a fraction. Examples of irrational numbers include $\pi, e, \sqrt{2}, \sqrt{3}$, and $\ln 2$. These numbers are widely used in geometry, such as π for circles, in calculus, in physical constants, and in modeling exponential growth or decay.

2.3 Properties of Real Numbers

The set of real numbers ($\mathbb{R}$) follows several fundamental rules that govern arithmetic operations, essential in algebra, calculus, and applied mathematics.

2.3.1 Closure

$\mathbb{R}$ is **closed under addition and multiplication**, meaning the sum or product of any two members is still a real number. Division by zero is the only exception.

2.3.2 Commutative

Addition and multiplication are **commutative**, so the order of numbers does not affect the result:

$$a + b = b + a \quad \text{and} \quad a \cdot b = b \cdot a.$$

2.3.3 Associative

Grouping of numbers does not change the outcome, reflecting the **associative rule** for both operations:

$$(a + b) + c = a + (b + c) \quad \text{and} \quad (a \cdot b) \cdot c = a \cdot (b \cdot c).$$

2.3.4 Distributive

Multiplication distributes over addition, meaning multiplying a number by a sum equals multiplying each term individually and then adding:

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

2.3.5 Identity

There exist **identities** for addition and multiplication. Adding 0 or multiplying by 1 leaves any number unchanged:

$$a + 0 = a \quad \text{and} \quad a \cdot 1 = a.$$

2.3.6 Inverse

Every number has **additive and multiplicative inverses** (except zero for multiplication). The additive inverse $-a$ satisfies $a + (-a) = 0$, and the reciprocal $\frac{1}{a}$ satisfies $a \cdot \frac{1}{a} = 1$.

2.4 Representation on Number Line

Figure Figure 2.2 illustrates a number line, a visual tool representing real numbers in order, which helps to clearly understand their relative positions and relationships.

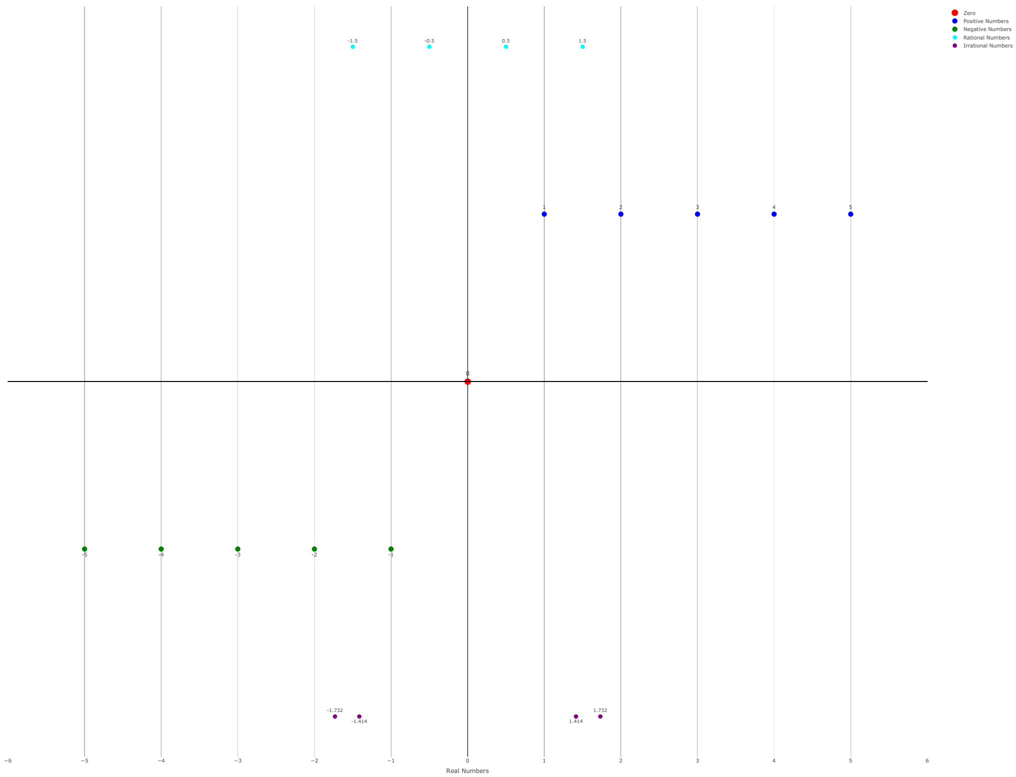

Figure 2.2: Representation on Number Line

The number line is a fundamental visual tool in mathematics that allows us to represent real numbers in order. It provides a clear way to understand the relative positions of numbers, including zero as the central reference point, positive numbers to the right, and negative numbers to the left. Rational numbers can be located precisely on the line, while irrational numbers occupy approximate positions between integers, filling in the gaps and illustrating the density of real numbers. The key concepts summarized in Table Table 2.2 highlight the main categories and their properties, helping to organize the understanding of real numbers on the number line.

Table 2.2: Key Concepts of Number Line

Concept	**Description**	**Notes**
Zero as Center	Central reference point separating positive and negative numbers.	Acts as reference for measuring distanc and direction (Figure 2.2)
Positive Numbers	Numbers greater than zero placed to the right of the origin; includes natural, whole, and positive fractions or decimals.	Magnitude increases to the right of zer
Negative Numbers	Numbers less than zero placed to the left of the origin; represents deficits, losses, or positions below reference.	Includes negative fractions and decima
Rational Numbers	Numbers expressible as fractions or terminating/repeating decimals; located exactly on the number line.	Each fraction corresponds to a precise point between integers
Irrational Numbers	Numbers not exactly expressible as fractions; approximate positions between integers filling in the gaps.	Examples: , $\sqrt{2}$, e; shows density of rea numbers

Table 2.3: Applications of Real Numbers with Examples

Domain	**Description**	**Examples**
Science & Engineering	Used to model measurements, physical quantities, and constants such as distance, mass, temperature, and speed; essential for calculations, simulations, and analyses in physics, chemistry, and engineering design.	Distance = 12.5 m, Temperature = Speed = 60 km/h
Economics	Represents prices, costs, profits, interest rates, and statistical data; allows precise modeling of financial transactions, economic trends, and optimization problems.	Price = $25.50, Profit = $1200, Int rate = 3.5%
Daily Life	Appears in counting money, measuring lengths or weights, telling time, and evaluating percentages; supports planning, cooking, shopping, and daily quantitative decision-making.	Buying 3 apples, Cooking 250 g flo Meeting at 14:30

2.5 Applications of Real Numbers

Real numbers play a crucial role in many fields because they can represent continuous quantities, perform precise measurements, and quantify relationships. Table Table 2.3 summarizes their main applications in science and engineering, economics, and everyday life.

Chapter 3

Essencials of Functions

Understanding **Functions** is fundamental in mathematics, as they describe the relationship between quantities and form the backbone of calculus, algebra, numerical modeling, and applied sciences. Functions allow us to model change, describe systems, and solve real-world problems [1]–[3].

The Figure 3.1 provides a **5W+1H mind map** for functions. This visualization guides learners through:

- **What** — definition, domain, range, and types of functions?.
- **Why** — importance and applications in math and science?.
- **When** — historical development and formalization?.
- **Where** — areas of application in engineering, physics, economics, and daily life?.
- **Who** — mathematicians and practitioners using functions?.
- **How** — representation through equations, tables, graphs, and intervals?.

Functions are one of the core concepts in mathematics, playing a central role in **modeling, analysis, and real-world applications**. Using the **5W+1H framework (What, Why, When, Where, Who, How)**, we can explore functions from multiple perspectives: their definition, importance, historical development, fields of application, key contributors, and different forms of representation.

Table Table 3.1 summarizes the key questions and provides illustrative examples of functions along with their interpretations for each 5W+1H category.

3.1 Definition

Functions are one of the core concepts in mathematics, playing a central role in **modeling, analysis, and real-world applications**. Using the **5W+1H framework (What, Why, When, Where, Who, How)**, we can explore functions from multiple perspectives: their definition, importance, historical development, fields of application, key contributors, and different forms of representation.

Table 3.1: 5W+1H Questions for Functions

	Description	Example_Function	Example_Output
What?			
What?	What is a function?	$f : X \to Y$	Each input $\to$ exactly one ou
What?	What are the domain and range?	$f(x) = x^2$, Domain = $\mathbb{R}$, Range = $[0, \infty)$	Domain all inputs; Range all outputs
What?	What types of functions exist?	Linear, Quadratic, Polynomial, Exponential, Trigonometric	Example: $f(x) = 2x + 1$, $f(x) = x^2 - 3$
What?	What are the key properties of functions?	Injective, Surjective, Bijective, Continuous, Monotone	e.g. $f(x) = x^2$ is not injective $\mathbb{R}$
Why?			
Why?	Why are functions important in mathematics?	Modeling $y = f(x)$ relationships	Predict outcomes, solve equat
Why?	Why do we need to understand function properties?	Analyzing $f(x)$ before applying to problems	Correct manipulation of $f(x)$
When?			
When?	When was the function concept formalized?	17th century (Leibniz, Euler)	Formalized in 1600s
When?	When are functions applied in real-life problems?	Finance: $A(t) = P(1 + r)^t$; Physics: $s(t) = v_0 t + \frac{1}{2}at^2$	Applications in simulations a modeling
Where?			
Where?	Where are functions used in science and engineering?	Ohm's law: $V = IR$, Newton's law: $F = ma$	Used in circuits, mechanics, chemistry
Where?	Where can functions be observed in economics and daily life?	Population growth $P(t) = P_0 e^{rt}$	Used in demand curves, budgeting
Who?			
Who?	Who were key mathematicians in developing function theory?	Euler, Leibniz, Dirichlet	Pioneers in function theory
Who?	Who uses functions in practical applications?	Scientists, engineers, economists	Real-world users across discip
How?			
How?	How are functions represented using equations?	$f(x) = x^2, f(x) = \sin x$	Symbolic form representation
How?	How are functions represented using tables?	Tabular form: $(x, f(x))$ pairs	Input-output lookup
How?	How are functions represented using graphs or intervals?	Graph of $f(x)$, interval $[a, b]$	Visual/geometric representati

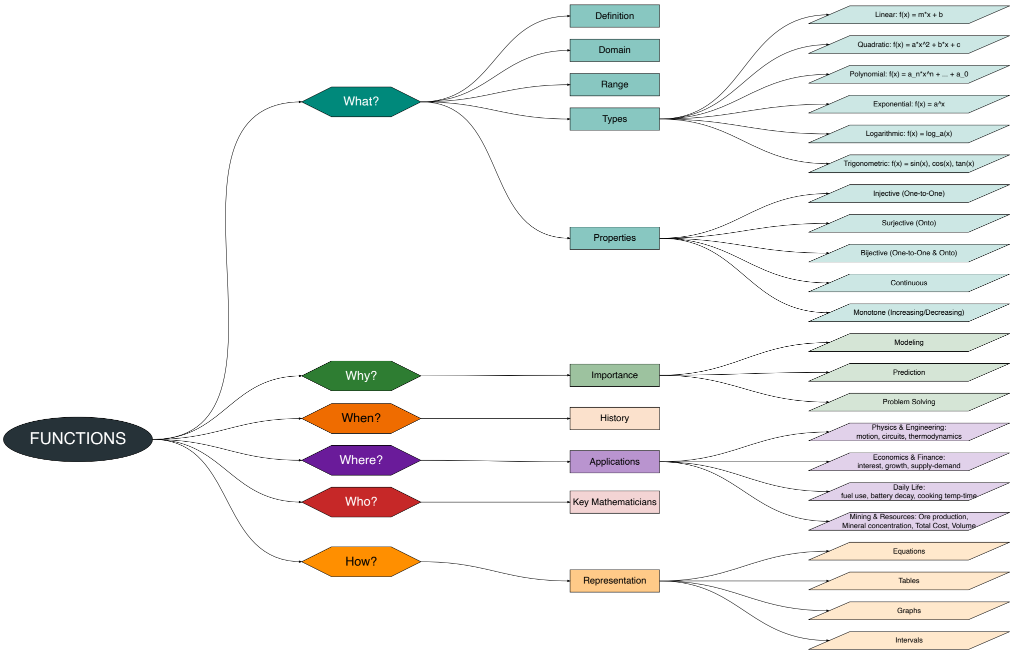

Figure 3.1: Detailed 5W+1H for Functions

A function f from set X to set Y is a rule that assigns **exactly one element of** Y to each element of X. Formally:

$$f : X \to Y \quad \text{such that } \forall x \in X, \exists! y \in Y \text{ with } y = f(x)$$

- **Domain**: the set of all inputs X
- **Range**: the set of all outputs Y

This definition ensures that every input has **one and only one output**, which distinguishes functions from more general relations. For example, consider $f(x) = x^2$ with domain $\mathbb{R}$. Each real number x is mapped to a single nonnegative real number $y = x^2$. In this case, the domain is $\mathbb{R}$ and the range is $\mathbb{R}_{\geq 0}$.

3.2 Types of Functions

Functions are fundamental tools in mathematics that describe the relationship between two quantities, typically denoted as an input x and an output $f(x)$. Each type of function has its own characteristics, shape, and application in real-world problems. Understanding these different types of functions is crucial not only in pure mathematics but also in various applied fields such as engineering, economics, physics, and mining engineering, where they are used to model growth, decay, oscillations, and relationships between variables.

Broadly, functions can be categorized into several groups, such as algebraic functions (linear, quadratic, polynomial), transcendental functions (exponential and logarithmic), and trigonometric functions (sine, cosine, tangent).

3.2.1 Algebraic

Algebraic functions (Figure Figure 3.2) are functions that can be expressed using a finite number of algebraic operations such as addition, subtraction, multiplication, division, and raising to a power. These functions form the foundation of many mathematical models and are widely applied in real-life problem solving.

The main types of algebraic functions include:

- **Linear Function**: $f(x) = mx + b$, which produce straight-line graphs and represent constant rates of change.
- **Quadratic Function**: $f(x) = ax^2 + bx + c$, which generate parabolic curves and are often used to model acceleration, projectile motion, or optimization problems.
- **Polynomial Function**: $f(x) = a_n x^n + \cdots + a_0$, which extend the idea of linear and quadratic functions to higher degrees, allowing the modeling of more complex relationships.

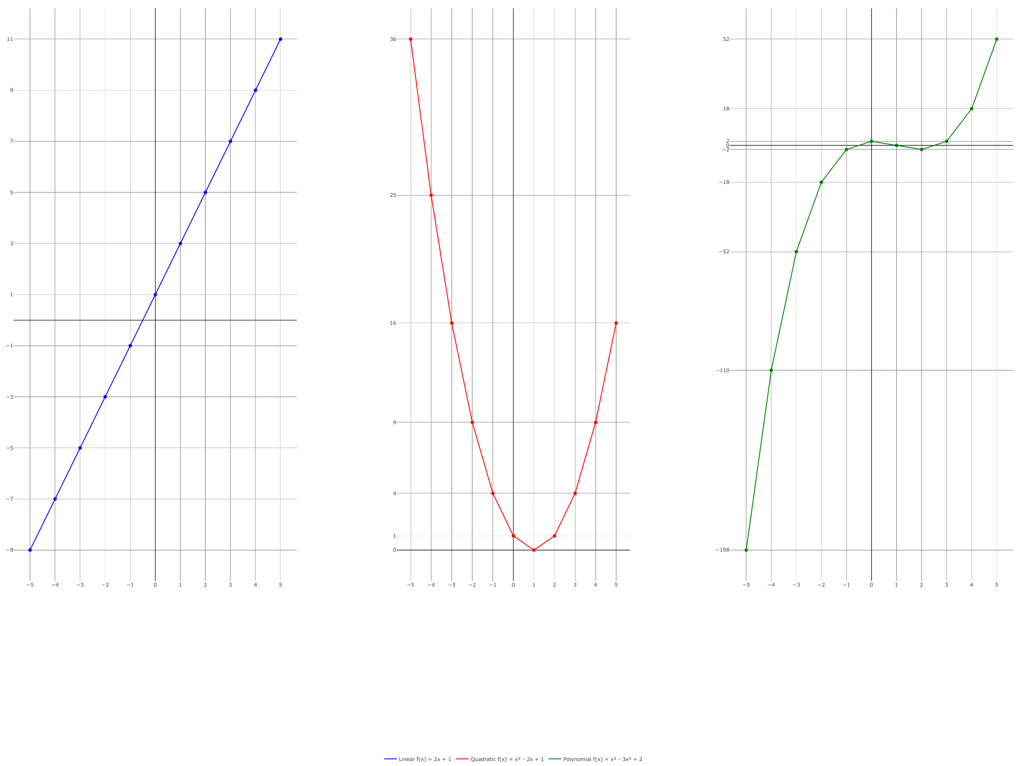

Figure 3.2: Algebraic Functions: Linear, Quadratic, and Polynomial (Side by Side)

3.2.2 Transcendental Functions

Transcendental functions (Figure Figure 3.3) are functions that cannot be expressed as finite combinations of algebraic operations. Unlike algebraic functions, they involve processes such as infinite series, exponentiation, and logarithms. These functions play a vital role in describing natural growth, decay, and scaling phenomena.

- **Exponential Function**: $f(x) = a^x$ are used to model rapid growth or decay, such as in population dynamics, radioactive decay, and compound interest.
- **Logarithmic Function**: $f(x) = \log_a x$ serve as the inverse of exponentials, commonly applied in measuring relative change, sound intensity (decibels), pH in chemistry, and data compression in computer science.

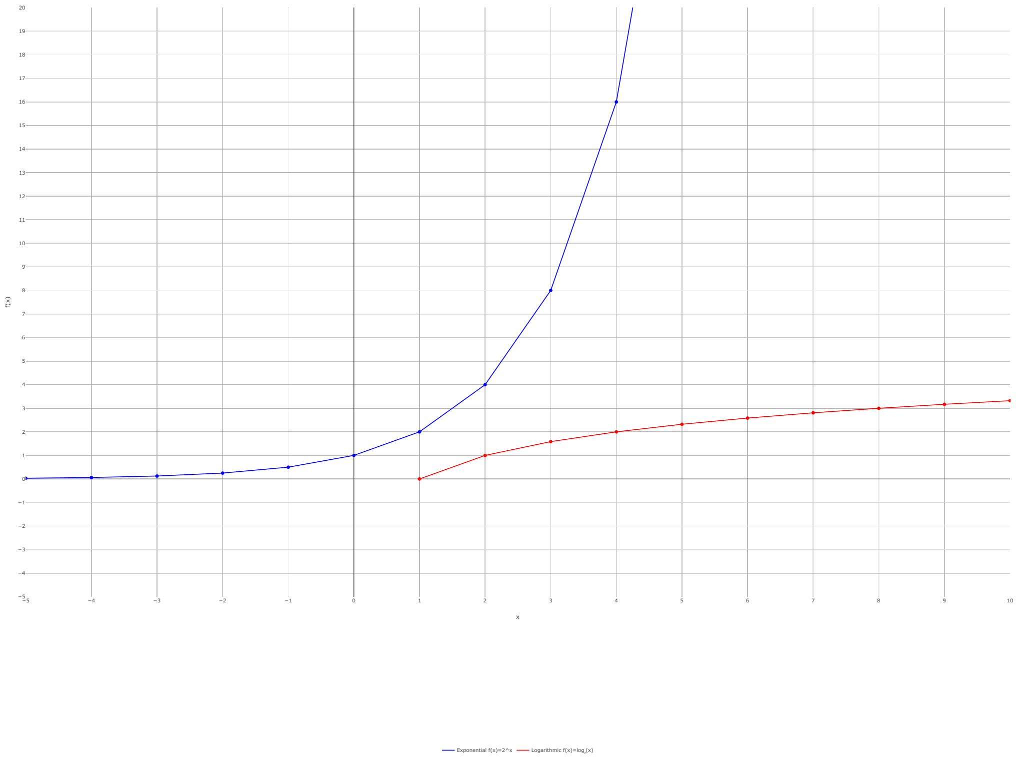

Figure 3.3: Exponential and Logarithmic Functions

3.2.3 Trigonometric Functions

The **trigonometric functions** $f(x) = \sin x, \cos x, \tan x$ describe the relationships between angles and the unit circle. They are fundamental in mathematics, physics, and engineering because they naturally model **oscillations, waves, and circular motion**. These functions are widely used in areas such as signal processing, alternating current circuits, sound and light waves, and applied fields like surveying and mining for modeling cyclic or repetitive patterns (see Figure Figure 3.4).

- **Sine** $(\sin x)$**:** Range $[-1, 1]$, period 2π, zeros at $0°, 180°, 360°$.
  Special values: $\sin 30° = \frac{1}{2}$, $\sin 45° = \frac{\sqrt{2}}{2}$, $\sin 60° = \frac{\sqrt{3}}{2}$
- **Cosine** $(\cos x)$**:** Range $[-1, 1]$, period 2π, zeros at $90°, 270°$.
  Special values: $\cos 30° = \frac{\sqrt{3}}{2}$, $\cos 45° = \frac{\sqrt{2}}{2}$, $\cos 60° = \frac{1}{2}$
- **Tangent** $(\tan x)$**:** Period π, undefined at $90°, 270°$.
  Special values: $\tan 30° = \frac{1}{\sqrt{3}}$, $\tan 45° = 1$, $\tan 60° = \sqrt{3}$

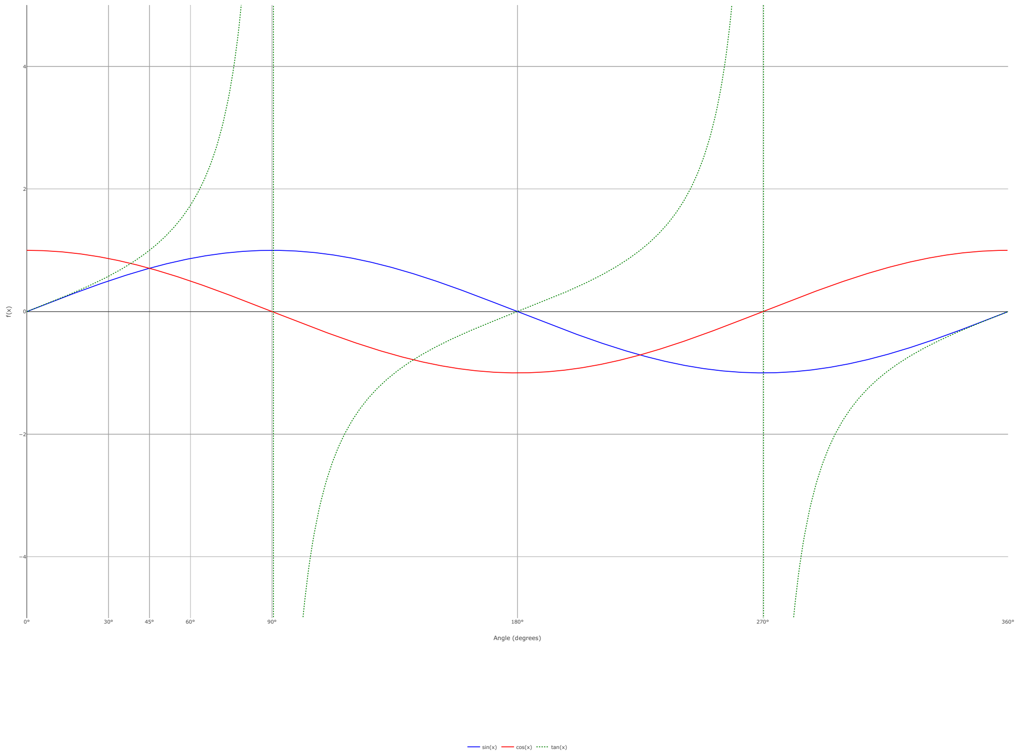

Figure 3.4: Trigonometric Functions: sin(x), cos(x), tan(x) with Special Angles

In trigonometry, certain angles are called special angles because their sine, cosine, and tangent values can be expressed in simple radical forms. These angles — such as $0°, 30°, 45°, 60°$, and $90°$ — are frequently used in mathematics, physics, and engineering for simplifying calculations.

3.3 Properties of Functions

Functions can be understood not only from their formulas, but also from the **properties** they possess. These properties describe how inputs and outputs are related, and how the function behaves across its domain and codomain (Figure Figure 3.5). Understanding these characteristics helps in identifying whether a function is one-to-one, onto, continuous, or monotone, which are fundamental concepts in both pure and applied mathematics.

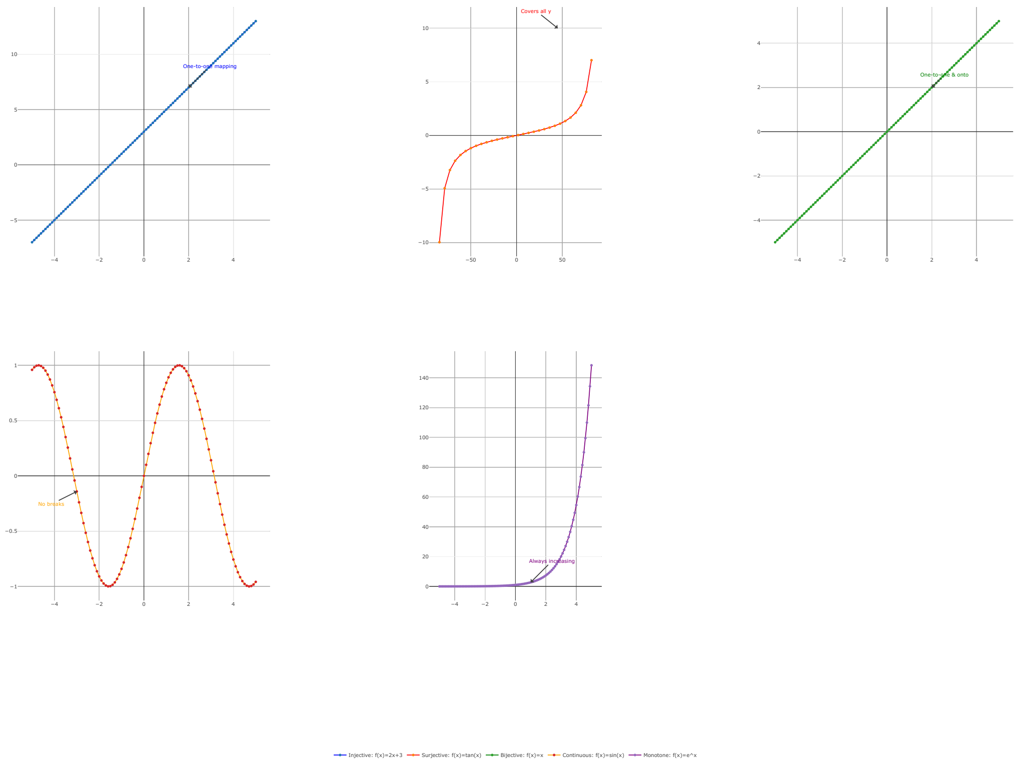

Figure 3.5: Function Properties: Injective, Surjective, Bijective, Continuous, Monotone

3.3.1 Injective (One-to-One)

A function is called **injective** if different inputs always produce different outputs. In other words, no two distinct values in the domain are mapped to the same value in the codomain. For example, $f(x) = 2x + 3$ is injective because every input corresponds to a unique output, whereas $f(x) = x^2$ is not injective over the real numbers, since both 2 and -2 map to the same value, 4. Understanding injective functions is important in applications where each output must correspond to a unique condition, such as tracking ore quality measurements in mining.

3.3.2 Surjective (Onto)

A function is **surjective** if every element in the codomain is "covered" by the function, meaning each possible output has at least one pre-image in the domain. For instance, $f(x) = x^3$ from $\mathbb{R}$ to $\mathbb{R}$ is surjective, while $f(x) = e^x$ is not surjective over all real numbers because it cannot produce negative values. Surjective functions are useful when it is essential that all potential outcomes are achievable, such as ensuring full coverage of production or resource allocation scenarios.

3.3.3 Bijective

When a function is both injective and surjective, it is **bijective**, establishing a perfect one-to-one correspondence between domain and codomain. Every output comes from exactly one input, and an inverse function always exists. For example, $f(x) = x + 5$ is bijective. In practice, bijective functions are valuable in simulations and data transformations, where each output needs to be traced back to a unique input without ambiguity.

3.3.4 Continuous

A function is **continuous** if its graph can be drawn without lifting the pen. Formally, f is continuous at $x = c$ if $\lim_{x \to c} f(x) = f(c)$. An example is $f(x) = \sin x$, continuous for all real numbers, while $f(x) = 1/x$ is discontinuous at $x = 0$. Continuity is crucial for modeling systems with predictable behavior, such as smooth motion of machinery or fluid flow in mining operations.

3.3.5 Monotone

Functions can also be **monotone**, consistently increasing or decreasing. A monotone increasing function ensures that larger inputs always produce larger outputs, while a monotone decreasing function produces smaller outputs for larger inputs. For example, $f(x) = 2x$ is monotone increasing, while $f(x) = -x$ is monotone decreasing. Monotone functions simplify analysis and optimization, for instance in predicting total ore extracted over time or planning production rates efficiently.

Table 3.2: Applications of Functions in Real Life and Mining

	Description	Example_Function	Example_Output
Science & Engineering			
Science & Engineering	Modeling position of moving object	$s(t) = 5t$	At t=3 s → s(3)=15 m
Science & Engineering	Modeling velocity of moving object	$v(t) = 2t$	At t=4 s → v(4)=8 m/s
Economics & Finance			
Economics & Finance	Supply-demand curves	$P(x) = 50 + 2x$	Selling 10 items → P(10)=70
Economics & Finance	Compound interest calculation	$A(t) = P(1 + r)^t$	Principal $1000, 5% annual, 3 years → $1157.63
Daily Life			
Daily Life	Temperature conversion	$F(C) = 9/5C + 32$	25°C → 77°F
Daily Life	Daily spending tracking	$S(d) = 10d + 5$	Day 7 → $75
Mining & Resources			
Mining & Resources	Ore production rate	$Q(t) = 1000 + 50t$	After 5 days → Q(5)=1250 tons
Mining & Resources	Mineral concentration	$C(x) = 0.8x + 5$	x=10 → C(10)=13%
Mining & Resources	Operational cost	$Cost(q) = 5000 + 20q$	Produce 100 units → Cost(100)=$7000
Mining & Resources	Heap volume	$V(A) = 2A + 100$	Area=50 m² → V(50)=200 m³

3.4 Applications of Functions

Chapter 4

Special Functions

Chapter 5

Limits of Functions

Chapter 6

Basic Derivatives

Chapter 7

Appllied of Derivatives

Chapter 8

Indefinite Integrals

Chapter 9

Appllied of Integrals

Transcendental Functions

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